

# Compact Linearization for Binary Quadratic Problems subject to Assignment Constraints

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November 4, 2016

## Abstract

We prove new necessary and sufficient conditions to carry out a compact linearization approach for a general class of binary quadratic problems subject to assignment constraints as it has been proposed by Liberti in 2007. The new conditions resolve inconsistencies that can occur when the original method is used. We also present a mixed-integer linear program to compute a minimally-sized linearization. When all the assignment constraints have non-overlapping variable support, this program is shown to have a totally unimodular constraint matrix. Finally, we give a polynomial-time combinatorial algorithm that is exact in this case and can still be used as a heuristic otherwise.

## 1 Introduction

In this paper, we are concerned with binary quadratic programs (BQPs) that comprise some assignment constraints over a set of  $\{0, 1\}$ -variables  $x_i, i \in N$  where  $N = \{1, \dots, n\}$  for a positive integer  $n$ . More precisely, let  $K$  be a set such that for each  $k \in K$  there is some index set  $A_k \subseteq N$  and such that exactly one of the variables  $x_i, i \in A_k$ , must attain the value 1. We assume that  $N \subseteq \{A_k \mid k \in K\}$ , i.e., the set of problem variables is covered by the union of the sets  $A_k$ . Additionally, bilinear terms  $y_{ij} = x_i x_j, i, j \in N$ , may occur in the objective function as well as in the set of constraints and are assumed to be collected in an ordered set  $E \subset V \times V$ . By commutativity, there is no loss of generality in requiring that  $i \leq j$  for each  $(i, j) \in E$ . Assuming an arbitrary set of  $m \geq 0$  further linear constraints  $Cx + Dy \geq b$  where  $C \in \mathbb{R}^{m \times n}$  and  $D \in \mathbb{R}^{m \times |E|}$ , the associated mixed-integer program discussed so far can be stated as follows:

$$\begin{aligned} \min \quad & c^T x + b^T y \\ \text{s.t.} \quad & \sum_{i \in A_k} x_i = 1 && \text{for all } k \in K \end{aligned} \tag{1}$$

$$\begin{aligned} & Cx + Dy \geq b \\ & y_{ij} = x_i x_j && \text{for all } (i, j) \in E \\ & x_i \in \{0, 1\} && \text{for all } i \in N \end{aligned} \tag{2}$$

The particular form studied here generalizes for example the quadratic assignment problem which is known to be NP-hard [23] as are BQPs with box constraints in general

[22], although there are some exceptions [3]. While there exist approaches to tackle BQPs directly, linearizations of quadratic and, more generally, polynomial programming problems, enable the application of well-studied mixed-integer linear programming techniques and have hence been an active field of research since the 1960s. This is also in the focus of this paper where we concentrate on the question how to realize constraints (2) for this particular type of problem by means of additional variables and additional linear constraints.

*Related Work.* The seminal idea to model binary conjunctions using additional (binary) variables is attributed to Fortet [8, 9] and discussed by Hammer and Rudeanu [16]. This method, that is also proposed in succeeding works by Balas [4], Zangwill [26] and Watters [25], and further discussed by Glover and Woolsey [13], requires two inequalities per linearization variable. Only shortly thereafter, Glover and Woolsey [14] found that the same effect can be achieved using *continuous* linearization variables when combining one of these two inequalities with two different ones. The outcome is a method that is until today regarded as the ‘standard linearization’ and where, in the binary quadratic context, each product  $x_i x_j$  is modeled using a variable  $y_{ij} \in [0, 1]$  and three constraints:

$$y_{ij} \leq x_i \tag{3}$$

$$y_{ij} \leq x_j \tag{4}$$

$$y_{ij} \geq x_i + x_j - 1 \tag{5}$$

Succeeding developments include a linearization technique without any additional variables but using a family of (exponentially many) inequalities by Balas and Mazzola [5]. Sherali and Adams showed how the introduction and subsequent linearization of additional nonlinear constraints can be used to obtain tighter linear programming (LP) relaxations for binary problems with (initially) linear constraints and a quadratic objective function [1]. This approach was later generalized in [2] to the so-called reformulation-linearization-technique (RLT). A single application of the RLT to the bounds constraints  $0 \leq x_i \leq 1$  of a binary program leads exactly to the above ‘standard linearization’ as proposed by Glover and Woolsey in [14].

Further linearization methods with more emphasis on problems where all nonlinearities appear only in the objective function are by Glover [12], Oral and Kettani [20, 21], Chaovalitwongse et al. [7], Sherali and Smith [24], Furini and Traversi [11], and, for general integer variables, by Billionnet et al. [6]. Specialized formulations for unconstrained binary quadratic programming problems have been given by Gueye and Michelon [15], and Hansen and Meyer [17].

For the particular binary quadratic problem as introduced at the beginning, Liberti developed a very elegant alternative *compact* linearization approach that exploits the structure imposed by the assignment constraints [18]. It can be seen as a very special application of the RLT that first determines, for each set  $A_k$ ,  $k \in K$ , another set  $B_k$  of original variables which are then multiplied with the according assignment constraint related to  $A_k$  yielding new additional equations. The choice of the sets  $B_k$  needs to be made such that the set of products  $F$  introduced this way covers the initial set of products  $E$ . Finally, the products in  $F$  are replaced by linearization variables. As already noted by Liberti, conceptually, this approach is in line with and a generalization of the strategy used by Frieze and Yadegar [10] for the quadratic assignment problem.

*Contribution.* In this paper, we show that one of the conditions originally specified as being necessary for the sets  $B_k$  to hold in order to yield a correct compact linearization is in fact neither necessary nor sufficient in every case. As a consequence, when

applying the original method to compute the sets  $B_k$ , inconsistent value assignments to the set of created variables can result. We reveal a new single necessary and sufficient condition to obtain a consistent linearization and prove its correctness. As a positive side effect, this condition can lead to smaller sets  $B_k$  and hence to a smaller number of necessary additional variables and constraints. In [18], also an integer program to compute minimum cardinality sets  $B_k$  (and hence a minimum number of additional equations) has been given. We present a similar mixed-integer linear program that establishes the new conditions and can now be used to minimize both, the number of created additional variables and additional constraints via a weighted objective function. Moreover, we show that the constraint matrix of this program is totally unimodular if all the sets  $A_k$ ,  $k \in K$ , are pairwise disjoint. Additionally, we provide an exact combinatorial and polynomial-time algorithm to compute optimal sets  $B_k$  in this case. With small modifications, the algorithm can also be used as a heuristic for the more general problem with overlapping sets  $A_k$ .

*Outline.* In Sect. 2, we review the compact linearization approach as developed in [18] and show that consistency of the linearization variables with their associated original variables is not implied by the conditions specified. New necessary conditions for a consistent linearization are characterized in Sect. 3 together with a correctness proof. Further, a mixed-integer linear program and a new combinatorial algorithm to compute compact linearizations are given. We close this paper with a conclusion and final remarks in Sect. 4.

## 2 Compact Linearization for Binary Quadratic Problems

The compact linearization approach for binary quadratic problems with assignment constraints by Liberti [18] is as follows: For each index set  $A_k$ , there is a corresponding index set  $B_k \subseteq N$  such that for each  $j \in B_k$  the assignment constraint (1) w.r.t.  $A_k$  is multiplied with  $x_j$ :

$$\sum_{i \in A_k} x_i x_j = x_j \quad \text{for all } j \in B_k, \text{ for all } k \in K \quad (6)$$

Each induced product  $x_i x_j$  is then replaced by a continuous linearization variable  $y_{ij}$  (if  $i \leq j$ ) or  $y_{ji}$  (otherwise). We denote the set of such created bilinear terms with  $F$  and we may again assume that  $i \leq j$  holds for each  $(i, j) \in F$ . More formally, Liberti defined the set  $F$  as

$$F = \{\phi(i, j) \mid (i, j) \in \bigcup_{k \in K} A_k \times B_k\}$$

where  $\phi(i, j) = (i, j)$  if  $i \leq j$  and  $\phi(i, j) = (j, i)$  otherwise. Using  $F$ , equations (6) can be rewritten as follows:

$$\sum_{i \in A_k, (i, j) \in F} y_{ij} + \sum_{i \in A_k, (j, i) \in F} y_{ji} = x_j \quad \text{for all } j \in B_k, \text{ for all } k \in K \quad (7)$$

The choice of the sets  $B_k$  is crucial for the correctness and the size of the resulting linearized problem formulation, as it directly determines the cardinality of the set  $F$  as well as the number of additional equations. On the one hand, the sets  $B_k$  must clearly be chosen such that  $F \supseteq E$ . On the other hand, this possibly (and in practice almost surely) involves the creation of additional linearization variables for some  $i \in A_k$  and  $j \in B_k$  where neither  $(i, j) \in E$  nor  $(j, i) \in E$ . Hence, the number of variables,  $|F|$ ,

will usually be larger than  $|E|$  but, as is discussed by Liberti and called *constraint-side compactness*, the number of equations can be considerably smaller than  $3|E|$  as it would be with the ‘standard’ approach and the formulation will still be at least as strong in terms of the LP relaxation of the problem.

This latter property can, e.g., be shown by arguing that solutions obeying all the equations (7) also satisfy the inequalities (3), (4), and (5) for all the variables introduced based on  $F$ . Since the added equations are all equations of the original problem multiplied by original variables, this also proves correctness of the linearization – no solutions feasible for the original problem can be excluded like this. Liberti follows exactly this strategy. His two main requirements for the choice of the sets is that the covering conditions  $E \subseteq F$  and  $A_k \subseteq B_k$  for all  $k \in K$  must be satisfied.

The condition  $E \subseteq F$  and the definition of  $F$  together imply that for each  $(i, j) \in E$  there must be some  $k \in K$  such that either  $i \in A_k$  and  $j \in B_k$  or  $j \in A_k$  and  $i \in B_k$  which finally establishes that  $(i, j) \in F$ . However, as we will show in the following, the condition  $A_k \subseteq B_k$  is not sufficient in every case in order to ensure that  $y_{ij} \leq x_j$  and  $y_{ij} \leq x_i$  simultaneously hold for all  $(i, j) \in F$ .

To see this, let  $k \in K$  be such that  $A_k \subsetneq B_k$ . We can assume without loss of generality that such a  $k$  exists since otherwise  $A_k = B_k$  must hold for all  $k \in K$  and this implies that  $E \subseteq F$  can only be established if, for all  $(i, j) \in E$ , there is an  $l \in K$  such that  $i, j \in A_l$ . In this case, however, all bilinear terms could be resolved trivially as  $y_{ij} = 0$  for all  $i \neq j$  and  $y_{ij} = x_i$  for all  $i = j$ . So let now  $j \in B_k \setminus A_k$ . In our linearization system, we obtain for  $j$  an equation:

$$\sum_{a \in A_k, (a, j) \in F} y_{aj} + \sum_{a \in A_k, (j, a) \in F} y_{ja} = x_j \quad (8)$$

Now fix any particular  $i = a \in A_k$  and assume, without loss of generality, that  $i < j$  ( $i = j$  is impossible since  $j \notin A_k$ ). Hence  $(i, j) \in F$  and equation (8) clearly establishes  $y_{ij} \leq x_j$ . The condition  $A_k \subseteq B_k$  now requires that there must be another equation for  $i$  of the form:

$$\sum_{a \in A_k, (a, i) \in F} y_{ai} + \sum_{a \in A_k, (i, a) \in F} y_{ia} = x_i \quad (9)$$

However, since  $j \notin A_k$ , the variable  $y_{ij}$  does not appear on the left hand side of (9). So if there is no other  $l \in K$ ,  $l \neq k$ , such that  $i \in B_l$  and  $j \in A_l$ , then there will be no equation that ever enforces  $y_{ij} \leq x_i$ .

The opposite case where, for some  $(i, j) \in F$ ,  $j \in A_k$  and  $i \in B_k \setminus A_k$  but there is no  $l \in K$ ,  $l \neq k$ , such that  $j \in B_l$  and  $i \in A_l$ , leads to the converse problem that there are equations that enforce  $y_{ij} \leq x_i$  but none that enforce  $y_{ij} \leq x_j$ .

Based on these observations, one can easily construct small examples where it holds that  $A_k \subseteq B_k$  for all  $k \in K$ , and  $E \subseteq F$ , but there exist  $(i, j) \in F$  for which inconsistent value assignments to  $y_{ij}$ ,  $x_i$  and  $x_j$  result. This remains particularly true when the originally proposed integer program is used to determine and minimize the total cardinality of the sets  $B_k$ .

### 3 Revised Compact Linearization

In the previous section, we have seen that the condition  $A_k \subseteq B_k$  for all  $k \in K$ , is *not sufficient* in every case to enforce that the inequalities  $y_{ij} \leq x_i$  and  $y_{ij} \leq x_j$  are satisfied for all  $(i, j) \in F$ . The discussion also already indicated that the two following conditions are *necessary* to enforce this.

**Condition 3.1.** For each  $(i, j) \in F$ , there is a  $k \in K$  such that  $i \in A_k$  and  $j \in B_k$ .

**Condition 3.2.** For each  $(i, j) \in F$ , there is an  $l \in K$  such that  $j \in A_l$  and  $i \in B_l$ .

For these two conditions, clearly,  $k = l$  is a valid choice. In this section, we will prove that these conditions are also *sufficient* in order to yield a correct linearization. This also means that the inclusion  $A_k \subseteq B_k$  is *not* a *necessary* condition.

**Theorem 3.3.** Let  $(i, j) \in F$ . If Conditions 3.1 and 3.2 are satisfied, then it holds that  $y_{ij} \leq x_i$ ,  $y_{ij} \leq x_j$  and  $y_{ij} \geq x_i + x_j - 1$ .

*Proof.* By Condition 3.1, there is a  $k \in K$  such that  $i \in A_k$ ,  $j \in B_k$  and hence the equation

$$\sum_{a \in A_k, (a, j) \in F} y_{aj} + \sum_{a \in A_k, (j, a) \in F} y_{ja} = x_j \quad (*)$$

exists and has  $y_{ij}$  on its left hand side. This establishes  $y_{ij} \leq x_j$ .

Similarly, by Condition 3.2, there is an  $l \in K$  such that  $j \in A_l$ ,  $i \in B_l$  and hence the equation

$$\sum_{a \in A_l, (a, i) \in F} y_{ai} + \sum_{a \in A_l, (i, a) \in F} y_{ia} = x_i \quad (**)$$

exists and has  $y_{ij}$  on its left hand side. This establishes  $y_{ij} \leq x_i$ .

As a consequence,  $y_{ij} = 0$  whenever  $x_i = 0$  or  $x_j = 0$ . In this case, the inequality  $y_{ij} \geq x_i + x_j - 1$  is trivially satisfied. Now let  $x_i = x_j = 1$ . Then the right hand sides of both  $(*)$  and  $(**)$  are equal to 1. The variable  $y_{ij}$  (is the only one that) occurs on the left hand sides of both of these equations. If  $y_{ij} = 1$ , this is consistent and correct. So suppose that  $y_{ij} < 1$  which implies that, in equation  $(*)$ , there is some  $y_{aj}$  (or  $y_{ja}$ ),  $a \neq i$ , with  $y_{aj} > 0$  ( $y_{ja} > 0$ ). Then, by the previous arguments and integrality of the  $x$ -variables,  $x_a = 1$ . This is however a contradiction to the assumption that  $x_i = 1$  as both  $i$  and  $a$  are contained in  $A_k$ .  $\square$

Minimum cardinality sets  $B_k$  (steering the number of additional equations) and minimum cardinality sets  $F$  (steering the number of additional variables) can be obtained using a mixed-integer program. The following one is a modification of the integer program in [18] in order to implement conditions 3.1 and 3.2, but even more importantly, to enforce them not only for  $(i, j) \in E$  but for  $(i, j) \in F$ .

$$\begin{aligned} \min \quad & w_{eqn} \left( \sum_{k \in K} \sum_{1 \leq i \leq n} z_{ik} \right) + w_{var} \left( \sum_{1 \leq i \leq n} \sum_{i \leq j \leq n} f_{ij} \right) \\ \text{s.t.} \quad & f_{ij} = 1 \quad \text{for all } (i, j) \in E \quad (10) \\ & f_{ij} \geq z_{jk} \quad \text{for all } k \in K, i \in A_k, j \in N, i \leq j \quad (11) \\ & f_{ji} \geq z_{jk} \quad \text{for all } k \in K, i \in A_k, j \in N, j < i \quad (12) \\ & \sum_{k: i \in A_k} z_{jk} \geq f_{ij} \quad \text{for all } 1 \leq i \leq j \leq n \quad (13) \\ & \sum_{k: j \in A_k} z_{ik} \geq f_{ij} \quad \text{for all } 1 \leq i \leq j \leq n \quad (14) \\ & f_{ij} \in [0, 1] \quad \text{for all } 1 \leq i \leq j \leq n \\ & z_{ik} \in \{0, 1\} \quad \text{for all } k \in K, 1 \leq i \leq n \end{aligned}$$

The formulation involves binary variables  $z_{ik}$  to be equal to 1 if  $i \in B_k$  and equal to zero otherwise. Further, to account for whether  $(i, j) \in F$ , there is a (continuous) variable  $f_{ij}$  for each  $1 \leq i \leq j \leq n$  that will be equal to 1 in this case and 0 otherwise. The

constraints (10) fix those  $f_{ij}$  to 1 where the corresponding pair  $(i, j)$  is contained in  $E$ . Further, whenever some  $j \in N$  is assigned to some  $B_k$ , then we need the corresponding variables  $(i, j) \in F$  or  $(j, i) \in F$  for all  $i \in A_k$  which is established by (11) and (12). Finally, if  $(i, j) \in F$ , then we require the two above conditions to be satisfied, namely that there is a  $k \in K$  such that  $i \in A_k$  and  $j \in B_k$  (13) and a (possibly different)  $k \in K$  such that  $j \in A_k$  and  $i \in B_k$  (14). Emphasis on either the number of created linearization variables or constraints can be given using the weights  $w_{var}$  and  $w_{eqn}$  in the objective function.

The underlying problem to be solved is a special two-stage covering problem whose complexity inherently depends on *how* the sets  $A_k, k \in K$ , cover the set  $N$ . In particular, if  $A_k \cap A_l = \emptyset$  for all  $k, l \in K, k \neq l$ , then the choices to be made in constraints (13) and (14) are unique and the problem can be solved as a linear program because the constraint matrix arising in this special case is totally unimodular (TU). To show this, we make use of the following lemma as stated in [19]:

**Lemma 3.4.** *If the  $\{-1, 0, 1\}$ -matrix  $A$  has no more than two nonzero entries in each column and if  $\sum_i a_{ij} = 0$  whenever column  $j$  has two nonzero coefficients, then  $A$  is TU.*

**Theorem 3.5.** *If  $A_k \cup A_l = \emptyset$ , for all  $k, l \in K, l \neq k$ , then the constraint matrix of the above mixed-integer program is TU.*

*Proof.* We interpret the constraint set (11)-(14) as the rows of a matrix  $A = [F \ Z]$  where  $F$  is the upper triangular matrix defined by  $\{f_{ij} \mid 1 \leq i \leq j \leq n\}$  and  $Z = (z_{ik})$  for  $i \in N$  and  $k \in K$ . Constraints (11) and (12) yield exactly one 1-entry in  $F$ , and exactly one  $-1$ -entry in  $Z$ . Conversely, constraints (13) and (14) yield exactly one  $-1$ -entry in  $F$ , and  $-$  since  $|\{k : i \in A_k\}| = 1$  for all  $i \in N$   $-$  exactly one 1-entry in  $Z$ . Hence, each row  $i$  of  $A$  has exactly two nonzero entries and  $\sum_j a_{ij} = 0$ . Moreover, the variable fixings (10) correspond to rows with only a single nonzero entry or can equivalently be interpreted as removing columns from  $A$  causing some of the other rows to have now less than two entries. Thus, by Lemma 3.4 and by the fact that the transpose of any TU matrix is TU,  $A$  is TU.  $\square$

Likewise, an exact solution can be obtained using a combinatorial algorithm (listed as Algorithm 1) where we assume that, for each  $i \in N$ , the unique index  $k : i \in A_k$  is given as  $K(i)$ . Basically, an initial set  $F_1 = E$  will then require some indices  $i$  to be assigned to some unique sets  $B_k$ . This may lead to further necessary  $y$ -variables yielding a set  $F_2 \supseteq F_1$  which in turn possibly requires further unique extensions of the sets  $B_k$  and so on until a steady state is reached. The asymptotic running time of this algorithm can be bounded by  $O(n^3)$ .

For the more general setting with overlapping  $A_k$ -sets and hence  $|\{k : i \in A_k\}| \geq 1$  for  $i \in N$ , the above proof of Theorem 3.5 fails and indeed, one can construct small artificial instances that have nonintegral optima. Nonetheless, the combinatorial algorithm can still be used when equipped with a routine to determine the indices  $k^*$  and  $l^*$ . One can follow, e.g., the following heuristic idea: To ease notation, let  $a(i, k) = 1$  if  $i \in A_k$  and  $a(i, k) = 0$  otherwise. Similarly, let  $b(i, k) = 1$  if  $i \in B_k$  and  $b(i, k) = 0$  otherwise. Adding some  $i$  to some  $B_k$  for the first time involves potentially creating additional variables  $(u, i)$  or  $(i, u) \in F$  for  $u \in A_k$ . More precisely, whether such a variable must be newly created depends on whether or not there already is some  $l \in K$  where  $u \in A_l$  and  $i \in B_l$  or  $i \in A_l$  and  $u \in B_l$ . Hence, the number of necessarily created variables when adding  $i$  to  $B_k$  is:  $c(i, k) = \sum_{u \in A_k} (\min_{l \in K} 1 - \max\{a(u, l)b(i, l), a(i, l)b(u, l)\})$ . So for a pair  $(i, j) \in F$ , we select  $k^* = \operatorname{argmin}_{k: i \in A_k} c(j, k)$  and  $l^* = \operatorname{argmin}_{k: j \in A_k} c(i, k)$ .

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**Algorithm 1** A Simple Algorithm to construct  $F$  and the sets  $B_k$  for all  $k \in K$ .

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function CONSTRUCTSETS(Sets  $E$ ,  $K$  and  $A_k$  for all  $k \in K$ )
  for all  $k \in K$  do
     $B_k \leftarrow \emptyset$ 
   $F \leftarrow E$ 
   $F_{new} \leftarrow E$ 
  while  $\emptyset \neq F_{add} \leftarrow \text{APPEND}(F, F_{new}, K, A_k, B_k)$  do
     $F \leftarrow F \cup F_{add}$ 
     $F_{new} \leftarrow F_{add}$ 
  return  $F$  and  $B_k$  for all  $k \in K$ 

function APPEND(Sets  $F$ ,  $F_{new}$ ,  $K$ ,  $A_k$  and  $B_k$  for all  $k \in K$ )
   $F_{add} \leftarrow \emptyset$ 
  for all  $(i, j) \in F_{new}$  do
     $k^* \leftarrow K(i)$ 
     $l^* \leftarrow K(j)$ 
    if  $j \notin B_{k^*}$  then
       $B_{k^*} \leftarrow B_{k^*} \cup \{j\}$ 
      for all  $a \in A_{k^*}$  do
        if  $a \leq j$  and  $(a, j) \notin F$  then
           $F_{add} \leftarrow F_{add} \cup \{(a, j)\}$ 
        else if  $a > j$  and  $(j, a) \notin F$  then
           $F_{add} \leftarrow F_{add} \cup \{(j, a)\}$ 
    if  $i \notin B_{l^*}$  then
       $B_{l^*} \leftarrow B_{l^*} \cup \{i\}$ 
      for all  $a \in A_{l^*}$  do
        if  $a \leq i$  and  $(a, i) \notin F$  then
           $F_{add} \leftarrow F_{add} \cup \{(a, i)\}$ 
        else if  $a > i$  and  $(i, a) \notin F$  then
           $F_{add} \leftarrow F_{add} \cup \{(i, a)\}$ 
  return  $F_{add}$ 

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Each such choice is a locally best one that neither respects the interdependences with any other choices nor the implications of the corresponding potentially induced new pairs  $(a, j)$  (or  $(j, a)$ ) for  $a \in A_{k^*}$  and  $(a, i)$  (or  $(i, a)$ ) for  $a \in A_{l^*}$ .

## 4 Conclusion and Final Remarks

In this paper, we introduced new necessary and sufficient conditions to apply the compact linearization approach for binary quadratic problems subject to assignment constraints as proposed by Liberti in [18]. These conditions are proven to lead to consistent value assignments for all linearization variables introduced. Further, a mixed-integer program has been presented that can be used to compute a linearization with the minimum number of additional variables and constraints. We also showed that, in the case where all the assignment constraints have non-overlapping variable support, this problem can be solved as a linear program as its constraint matrix is totally unimodular. Alternatively, an exact polynomial-time combinatorial algorithm is proposed that can also be used in a heuristic fashion for the more general setting with overlapping variable sets in the assignment constraints.



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